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Errors in Lattice Extractions of α_s Due to Use of Unphysical Pion Masses

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Abstract

We investigate the errors due to the use of unphysical values of light quark masses in lattice extractions of α_s . A functional form for the pion mass dependence of the quarkonium mass splittings (Δm) is given as an expansion in $m_\pi/(4\pi f_\pi)$ and $m_\pi r_B$, where r_B is the quarkonium Bohr radius. We find that, to lowest order, $\Delta m \simeq A + Bm_\pi^2$, where the scale of B is given by $f_\pi^2 r_B^3$. To order m_π^4 there are four unknown coefficients, however, utilizing multipole and operator product expansions, symmetry arguments eliminate one of the four unknowns. Using the central values for the lattice spacings which were extracted using two different, unphysical values for the pion mass, we find that the errors introduced by extrapolating to the physical regime are comparable to the errors quoted due to other sources. Extrapolation to physical values of the pion mass *increases* the value of $\alpha_s(M_Z)$, bringing its value closer to the high energy extractions.

1 Introduction

Precision measurements of α_s are motivated not only by our need for more precise predictions within the confines of perturbative QCD, but also by the possibility that such a measurement could lead to hints of new physics. For instance, GUTS often make predictions for the value of α_s at low energies, and thus, these theories may be tested via a precise determination of α_s . Presently, there seems to be a slight discrepancy between low and high energy extractions of α_s [1], and it has become crucial to have a good handle on the errors involved before we can determine whether or not this discrepancy is a signal for new physics. One of the low energy extractions, and the one with the smallest quoted error bar, is performed via a measurement of the splittings between quarkonia levels on the lattice.

There have been several lattice extractions of $\alpha_s(M_Z)$ [2] [3], with the smallest quoted error being given in [2]. The extractions are performed by determining the lattice spacing via a measurement of the splittings between different quarkonia energy levels, the values of which are known from laboratory experiments. Unfortunately, due to computational difficulties, the measurements are performed in a theory with unphysical values for the masses of the light mesons. The values of α_s extracted in these unphysical theories are then assumed to differ little from the physical values. The justification for this assumption is that as long as the light quark masses are small compared the typical momenta in the bound state, the level splittings should be insensitive to the quark masses.

In this note we will determine the size of these errors in a systematic expansion in the pion mass. More specifically, we address the errors incurred by extrapolation to physical values of the light quark masses and the assumption of $SU(3)$ flavor symmetry. We use previously developed techniques [4][5][6] to determine an appropriate effective field theory to describe the interaction of light mesons with heavy $Q\bar{Q}$ bound states. Once the correct effective field theory is found we are able to calculate the errors introduced due to extrapolation to physical values with no guess work on unknown functional forms.

The remainder of this paper is structured as follows. In the next section we will review the lattice measurements with emphasis on the approximations used. In section three we

present our results of the errors. In section four we write down the chiral Lagrangian which describes the interaction of the quarkonia with the light degrees of freedom. In section five we show that, within the confines of a multipole expansion, it is possible to map the theory of heavy quarks onto a chiral Lagrangian using symmetry arguments first noted in [12]. The final section is reserved for conclusions and the need for further computations.

2 Present Lattice Extractions

The accuracy of present lattice calculations are limited by the inability to properly simulate the light quarks. In general, it is difficult enough to include dynamical fermions in the calculations, no less calculate using their physical masses. Thus, to measure physical parameters with accuracies which are competitive with experiment, it is necessary to look for observables whose values are not strongly dependent on the dynamics of the light quarks. This is one of the primary reasons why heavy quark anti-quark bound states have been chosen as laboratories for measuring the value of α_s . The level splittings in quarkonia are expected to be insensitive to the masses and the number of light quarks due to the fact that the Bohr radius is so small compared to the hadronic scale $1/\Lambda_{QCD}$. It is therefore hoped that calculating with unphysical masses for the light mesons should be a very good approximation, and the computational problems which arise in the physical region of parameter space can be avoided.

In this section we review the method by which the measurements are made following the work of ref [2]. There are several sources of errors in these calculations, all of which are claimed to be under control. Here we will focus only on those errors induced by extrapolation to physical values of the light quark masses.

The coupling is measured by first determining the lattice spacing, a , in a theory with some unphysical value for the pion and kaon masses. This is done by measuring the level splittings in quarkonia and then extracting the value of a by assuming that the value of the splitting in this unphysical theory is very close to physical value. Once the lattice spacing is

Table I: Measurements of $\alpha_P(8.2 \text{ GeV})$.

	1S-2S	1-S-1P
$am_q = .01, n_f = 2, am_\pi = 0.419(2)$	0.1793(16)	0.1777(23)
$am_q = .025, n_f = 2, am_\pi = 0.269(1)$	0.1760(35)	0.1735(28)
$n_f = 0$	0.1551(11)	0.1505(9)

known, the coupling α_P is determined by its plaquette value.

$$-\ln W_{1,1} = \frac{4\pi}{3} \alpha_P \left(\frac{3.41}{a} \right) [1 - (1.185 + .070n_f)\alpha_P]. \quad (1)$$

The value of α_s in the \overline{MS} is then determined through the relation

$$\alpha_{\overline{MS}}^{n_f}(Q) = \alpha_P^{n_f}(e^{5/6}Q) \left[1 + \frac{2}{\pi} \alpha_P^{n_f} + O(\alpha_P^{n_f})^2 \right]. \quad (2)$$

The measurements are performed with varying numbers (up to $n_f = 2$) of light quarks with several different, unphysical quark masses. The latest published results from the NRQCD collaboration are shown in table I[2]. The physical values of α_s are then found by extrapolating to $n_f = 3$ using the fact that the inverse coupling $1/\alpha_P^{(n_f)}$ is known to be almost linear for small changes in n_f . The assumption is made that the results are independent of the pion mass and $SU(3)$ symmetry breaking effects are ignored. No errors are quoted for these parts of the procedure.

The value of the coupling at the scale M_Z is then determined by first running down to the mass of the charm quark using the three flavor beta function and then running back up to M_Z , taking into account the b quark threshold. Errors due to variations in the thresholds were shown to be negligible. Using $m_q a = .01$ the values found were:

$$\alpha_{\overline{MS}}(M_Z) = 0.1152(24) \quad 1S - 1P \quad (3)$$

$$\alpha_{\overline{MS}}(M_Z) = 0.1154(26) \quad 1S - 2S. \quad (4)$$

The errors quoted here were found assuming a coefficient of one for the α_P^2 term in (2).

3 Results: Procedure for Calculating the Errors

We will now present the results for the errors which will be derived in a later section. In a QCD like theory with an arbitrary pion mass, the dimensionless mass splitting can be

written as

$$\begin{aligned}
a\Delta m &= aA + \frac{B}{a^3} f^2 \left[\frac{1}{(4\pi f)^2} \left((am_\eta)^4 \log \frac{m_\eta^2}{4\pi\mu^2} + 3(am_\pi)^4 \log \frac{m_\pi^2}{4\pi\mu^2} + 4(am_K)^4 \log \frac{m_K^2}{4\pi\mu^2} \right) \right. \\
&\quad \left. - 6C((am_K)^2 + \frac{(am_\pi)^2}{2}) \right] + \frac{D(\mu)}{a^3 16\pi^2} (3(am_\pi)^4 + 4(am_K)^4 + (am_\eta)^4) + \dots \quad (5)
\end{aligned}$$

Here a is the lattice spacing, A, B, C, D are dimensionful unknowns, and the higher order terms left out are suppressed by powers of $\frac{m_\pi^2}{4\pi f_\pi}$. We will show in a later section that within the confines of a multipole expansion, the coefficient C is 1 at leading order under certain conditions. Furthermore, in general, it is possible to calculate B using potential models, but we will leave it as an unknown for now. The parameter f is independent of m_π , and its value can be calculated in terms of m_π and f_π in QCD. In a given theory with light quark mass m_q , we know only the dimensionless combination am_π . In the two theories which have been investigated to date $am_\pi = 0.269(1), 0.419(2)$ for $am_q = 0.01, 0.025$ respectively.

Furthermore, all the simulations which have been performed with dynamical fermions have only included two light quark flavors. We therefore reduce the general form (5) to the case of $SU(2)$

$$a\Delta m = \left[aA_2 + \frac{B_2}{a} f^2 (am_\pi)^2 \left(-2 + \frac{(am_\pi)^2}{(af)^2} \frac{1}{16\pi^2} \log \frac{m_\pi^2}{4\pi\mu^2} \right) + \frac{D_2(\mu)}{16\pi^2 a^3} (am_\pi)^4 \right]. \quad (6)$$

In the future, when simulations are performed with three light quarks, we will be able to determine the corrections due to $SU(3)$ breaking using (5). Notice that going to the $SU(2)$ case does not reduce the number of unknowns.

Since we have only two data points to fit, let us begin by considering the corrections only up to $O(m_\pi^2)$. In this case, we may solve for A and B and subsequently determine the error induced by extrapolating to physical masses for the pions. For the case of three flavors one should choose the value $(m_\pi^2 + 2m_K^2)/3$ for the “physical pion mass”. Given that in the extraction process we find α_P with two flavors then extrapolate to three, we see no reason why this should remain the correct value. Since we are calculating errors here, we believe that the proper choice should be somewhere between the 140 and 410 MeV.

Using the physical values $\frac{\Delta m_{1S-1P}}{m_\pi} = 2.86$ and $\frac{\Delta m_{1S-2S}}{m_\pi} = 4.00$ for the $1S-1P$ and $2S-1S$

Table II: Measurements of $\alpha_{\overline{MS}}(91.2 \text{ GeV})$.

$m_\pi \text{ (MeV)}$	1S-2S	1S-1P
910	0.1134	0.1149
640	0.1154(26)	0.1152(24)
410	0.1174	0.1167
140	0.1184	0.1174

splittings in the Upsilon system respectively, we find for $m_\pi = 140(410) \text{ MeV}$

$$a^{-1} |_{phys} = 2.56 \text{ (2.47)} \quad 1S - 1P \quad (7)$$

$$a^{-1} |_{phys} = 2.53 \text{ (2.52)} \quad 1S - 2S. \quad (8)$$

The authors of [2] found the value $\alpha_P(8.32) = 0.178 \text{ (1S-1P)}$ and $\alpha_P(8.08) = 0.179 \text{ (1S-2S)}$ for $m_\pi = 640 \text{ MeV}$, leading to, up to quartic terms in the pion mass, $\alpha_P(8.73) = 0.178 \text{ (1S-1P)}$ and $\alpha_P(8.63) = 0.179 \text{ (1S-2S)}$, when the pion mass takes on its physical value. We have assumed here that $\ln W_{1,1}$ is independent of the pion mass given that it is a short distance quantity. Varying the pion mass was shown to change α_P by less than 0.2% [8]. Table II shows our results for the dependence of $\alpha_{\overline{MS}}(M_Z)$ on the value of the pion mass.

In calculating the size of the error here we must consider the fact that we have assumed that the expansion (5) is well behaved. Indeed, in calculating the values of the coefficients we have used the data for a theory in which the pion mass is on the order $\sim 900 \text{ MeV}$, therefore these results should not be trusted at a quantitative level but should be a qualitative estimate of the error. To get an accurate value for the constants A_2, B_2 and D_2 it is important that at least one other lattice simulation be performed with light quark masses smaller than $am_q = 0.025$.

We may make the guess that the non-analytic piece dominates the counter-term as is sometimes done when working with chiral Lagrangians. Taking $4\pi\mu^2$ to be the lattice spacing, we find that the results shift little. The net effect of the logs is to decrease the value of $\alpha_s(M_Z)$ at the level of 0.5%, which is to be compared to the 3% effect found when just the piece quadratic in the pion mass is kept.

4 Chiral Lagrangians and Heavy Quarks

Chiral Lagrangians for heavy-light systems have been utilized for several years [10]. For these systems there is only one scale in the theory besides the hadronic scale, namely the mass of the heavy quark. This scale is trivially removed by rescaling the mesonic field as was originally done in heavy quark effective field theory. One redefines the heavy meson field according to

$$H(x) = e^{-im_Q v \cdot x} H^v(x) \quad (9)$$

and treats this field as a classical static source labeled by its velocity, v . In so doing, spin symmetry as well as flavor symmetry (between D and B mesons) becomes manifest[11]. Once the heavy quark mass has been scaled out, there is no question as to how the operators scale.

In the case of heavy-heavy systems, chiral symmetry has been utilized in decays for quite sometime [12][13][14][15]. The chiral Lagrangian for these systems, while equivalent to the current algebra approach utilized in the above references, organizes the expansion perhaps more naturally. Such Lagrangian have only been written down in the literature more recently [16][17]. Though the heavy-heavy system does not posses a flavor symmetry, the spin symmetry remains, contrary to what is claimed in [16]. Spin symmetry breaking effects are suppressed by powers of the relative velocity of the heavy quarks instead of inverse powers of the mass as in the case of the heavy-light systems. The velocity scaling rules for the factorized heavy quark matrix elements will be the same as those derived for the NRQCD formalism [18]. For our calculation, the spin symmetry will not be relevant and therefore it will not be manifested in our phenomenological Lagrangian.

Let us consider the chiral Lagrangian for the spin one $1S$ and $2S$ states. For the $1S$ state the lowest order Lagrangian, which is invariant under chiral symmetry is given by

$$\begin{aligned} L_{int} = & c_1 h^{(v)\mu} h_\mu^{(v)*} Tr(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + c_2 h_\mu^{(v)} h_\nu^{(v)*} Tr(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma + \partial^\nu \Sigma^\dagger \partial^\mu \Sigma) \\ & + c_3 h^{(v)\mu} h_\mu^{(v)*} Tr(v \cdot \partial \Sigma^\dagger v \cdot \partial \Sigma), \end{aligned} \quad (10)$$

and similarly for the $2S$ state. Here, and throughout the rest of this paper, we follow the

notation and normalizations in [20]. Σ is a unitary 3×3 matrix which contains Goldstone octet fields. $c_{1,2,3}$ are dimensionful parameters whose scale at this point is unknown but will be determined in the next section. h_μ^v is the spin one heavy quarkonia field, labeled by its velocity v , and satisfies

$$iv \cdot \partial h_\mu^{(v)} = 0, \quad v^\mu h_\mu^{(v)} = 0. \quad (11)$$

We will ignore terms which are off diagonal in the quarkonia fields as they will give subleading contributions to the shift in the level splitting. It is possible to make the spin symmetry manifest by including the η state along with the vector $1S$ state as is done for the heavy-light system, but for our purposes this is unnecessary. The leading order chiral symmetry breaking piece of the Lagrangian is

$$L_{\chi sb} = c_4 h^{(v)\mu} h_\mu^{(v)*} Tr(M(\Sigma + \Sigma^\dagger)), \quad (12)$$

and M is the diagonal quark mass matrix. As will be shown below, higher dimension terms in both the chirally symmetric as well as chiral symmetry breaking pieces of the Lagrangian will be suppressed by powers of $4\pi f$.

4.1 Multipole and Twist Expansions

It was pointed out by Gottfried [19] that interactions of long wavelength gluons with quarkonia should be well described by a multipole expansion, which yields an expansion in $E/(r_B^{-1})$ where E is the external gluonic energy scale and r_B is the Bohr radius. For decay processes this leads to an expansion in the relative quark velocity v . We are interested in the light quark mass dependence of the mass splittings, and therefore the expansion parameter becomes $m_\pi/(r_B^{-1})$. To implement the expansion we assume that the decay goes through a two step process. Which is to say, the hadronization process factorizes from the decay process. Thus, we split the Hamiltonian up as follows

$$H = H_Q + H_g + H_{int}, \quad (13)$$

where H_Q acts on the heavy quarks only and includes the attractive and repulsive Coulomb potentials for the singlet and octet states respectively. H_g acts on the gluonic degrees of

freedom, and H_{int} describes the interaction between the quarkonium and the light degrees of freedom. We treat H_{int} as a perturbation and the eigenstates of the leading order Hamiltonian are of the factorized form

$$|\psi\rangle = |\phi\rangle |G\rangle. \quad (14)$$

$|G\rangle$ corresponds to the state of the dynamical gluons and is the ground state $|0\rangle$ when the quarks are in a relative color singlet state. The coupling to the spin will be higher order in a velocity expansion.

The calculation of the matrix element for two gluon emission then goes through much in the same way as the well known calculation of Rayleigh scattering in QED. Here, we shall simply state the result for the Euclidean space amplitude and refer the reader to [4][5] for details.

$$M = -\frac{g^2}{2N} \langle \phi_s | \vec{r} \cdot \vec{E}^a \frac{1}{H_a - \epsilon - D^0} \vec{r} \cdot \vec{E}^a | \phi_s \rangle, \quad (15)$$

The wave function of the state in which the heavy quarks are in a relative color singlet state, ϕ_s , is an eigenstate of the Hamiltonian containing the singlet part of the Coulomb potential and, D_0 acts on the gluonic Hilbert space. The factor $1/(H_a - \epsilon - D^0)$ accounts for adjoint-state propagation between gluon emissions. ϵ is the eigenenergy of ϕ , and H_a is the repulsive adjoint Coulomb Hamiltonian acting on the quark piece of the Hilbert space. This so called “double-dipole” amplitude is the leading term in the multipole expansions.

The fact that the chromo-electric field \vec{E}^a no longer depends upon the relative separation of the quarks and anti-quark allows us to write this amplitude in an OPE-like form where all the details of the bound state are in the Wilson coefficient \tilde{C}_n . For S wave states we may write

$$M = - \sum_{n=2, \text{ even}}^{\infty} \tilde{C}_n \epsilon_0^{2-n} r_B^3 \left[\frac{1}{2} \vec{E}^a (D^0)^{n-2} \vec{E}^a \right], \quad (16)$$

where

$$\tilde{C}_n = \frac{16\pi}{N^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{3} \left| \frac{\vec{r}}{r_B} \psi \right|^2 \frac{1}{[H_a/\epsilon_0 + \epsilon/\epsilon_0]^{n-1}}, \quad (17)$$

$$r_B = \frac{16\pi}{g^2 N m_Q}, \quad \epsilon_0 = \left(\frac{g^2 N}{16\pi} \right)^2 m_Q, \quad (18)$$

and we have neglected spin symmetry breaking corrections. We may therefore reproduce the interactions of long wavelength gluons with the quarkonia via a (Minkowski) interaction Lagrangian

$$L_{int} = \sum_{n=2 \text{ even}}^{\infty} \tilde{C}_n \epsilon_0^{2-n} r_B^3 h^{(v)\mu} h_\mu^{(v)*} \left(\frac{1}{2} \vec{E} \cdot (iD^0)^{n-2} \vec{E} \right) + \dots \quad (19)$$

where the ellipses denote higher multipoles as well as higher orders in α_s . This sum looks much like the leading twist expansion in deep-inelastic scattering, except the corrections here are not power suppressed. Thus we will refer to the expansion in the number of field insertions (read coupling expansion) as the twist expansion¹.

If one assumes that the leading term in (19) dominates, then it is possible to map this effective Lagrangian onto a chiral Lagrangian using symmetry arguments, as will be done in the next section. However, there is no a priori reason why this should be a good approximation. For decay processes, this expansion may not be useful since the radiated gluon energies are the mass splittings (of the order the Rydberg). Thus, as was pointed out in [21], previous attempts at using this expansion to calculate higher order corrections to quark mass relations [23] are on dubious ground. However, there is phenomenological evidence to the effect that keeping only the leading term in (19) may not be unreasonable for the case of decays (see e.g. [12]). For the case of interest here we are concerned with self interactions and the OPE is a systematic expansion in $\Lambda_{\text{QCD}}/\epsilon_0$. We will come back to a detailed analysis regarding this issue in a future publication.

5 Mapping Onto the Chiral Lagrangian

It was pointed on in [12] that if a given double-dipole transition is dominated by the leading term in the OPE (16), then it is possible to map this interaction onto a chiral Lagrangian, thus allowing us to reduce the number of unknown parameters. Matrix elements of the double electric dipole operator $\alpha_s \vec{E}^a \cdot \vec{E}^a$, or the electric-magnetic dipole interference operator,

¹This similarity was utilized in [5].

$\alpha_s \vec{E}^a \cdot \vec{H}^a$ [15], can be mapped to a chiral Lagrangian by using the fact that

$$\partial_\mu j_D^\mu = T_\mu^\mu = \frac{1}{2} \frac{\beta(g)}{g} G_{\mu\nu}^a G^{\mu\nu a} + \sum_i (1 - \gamma_i) m_i \bar{\Psi}_i \Psi_i. \quad (20)$$

and

$$i\partial_\mu j_5^\mu = \frac{3\alpha_s}{8\pi} G_{\mu\nu}^a \tilde{G}^{\mu\nu a} + \sum_i 2m_i \bar{\Psi}_i \gamma_5 \Psi_i. \quad (21)$$

In the above equations, the sum is taken over the three light quark flavors, j_5^μ is the axial current, j_D^μ is the dilatation current and $T_{\mu\nu}$ is the stress energy tensor.

We see that knowing the symmetry transformation properties of the Goldstone boson fields in the chiral Lagrangian allows us to determine how the quarkonia couple to the light degrees of freedom. This will allow us to disentangle the many scales which arise in the effective field theory. Furthermore, both operators in (20) are renormalization group invariants[22], and thus we need not worry about at which scale to choose the value for the strong coupling.

Consider the case where the coupling to pions is dominated by the leading term in the OPE of the double-dipole approximation, namely \vec{E}^2 . How does this operator map into the chiral Lagrangian? Following (20) we calculate the trace of the stress-energy tensor for the chiral Lagrangian which is given to order m_π^2 in a chiral expansion by

$$\partial_\mu j_D^\mu = T_\mu^\mu = -\frac{f^2}{2} \text{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) - 2f^2 \mu \text{Tr}(m\Sigma + m\Sigma^\dagger) + \dots \quad (22)$$

Here the terms which are left off are down by powers of $4\pi f$. Thus, given the analysis in the previous section, we now know that the lowest order coupling between the quarkonium state and the light degrees of freedom is given by

$$L_{int} = -\frac{g}{2\beta(g)} \tilde{C}_2 r_B^3 h^{(v)*} \cdot h^{(v)} \left(\partial_\mu j_D^\mu - \sum_i (1 - \gamma_i) m_i \bar{\Psi}_i \Psi_i \right). \quad (23)$$

Here γ_i is the anomalous dimension for the quark field Ψ_i . We have neglected the term in the relation between $\vec{E}^a \cdot \vec{E}^a$ and $G_{\mu\nu} G^{\mu\nu}$ which involves the magnetic field as it is suppressed by spin symmetry.

We may now argue that the term coming from the anomalous dimensions is small compared to one based on the idea that it has arisen from integrating out shorter wavelengths

and should thus be suppressed by powers of the strong coupling evaluated at a perturbative scale. If we drop this term then we are left with the following coupling

$$L_{h^*h\pi} = \tilde{C}_2 r_B^3 \frac{g}{2\beta(g)} h^{(v)*} \cdot h^{(v)} \left(f^2 \text{Tr}(\partial_\mu \Sigma) (\partial^\mu \Sigma^\dagger) + 3f^2 \mu \text{Tr}(m\Sigma + m\Sigma^\dagger) \right) + \dots \quad (24)$$

If we are not willing to assume that the piece coming from the anomalous dimension is small, then the relative sizes of the two terms in (24) becomes incalculable and we must include a new unknown parameter into the Lagrangian. Notice that the assumption that the first term in the OPE of the leading twist sum dominates effectively reduces the number of possible terms in the Lagrangian at this order. For instance the terms

$$f^2 \bar{h}_\mu^v h_\nu^v \text{Tr} \partial^\mu \Sigma \partial^\nu \Sigma^\dagger, \quad \bar{h}_v \cdot h_v f^2 \text{Tr}(v \cdot \partial) \Sigma (v \cdot \partial) \Sigma^\dagger \quad (25)$$

are higher order in the OPE.

5.1 Derivation of mass splittings formula

We may now use (24) to calculate the pion mass dependence on the level splitting. The splitting will get a contribution from a tree level counter-term which is independent of the pion mass, and the leading dependence from the pion mass will come from the second term in (24). At order $O(m_\pi^4)$ there will be a one loop correction which will contribute a piece which is non-analytic in the pion mass as well as an unknown counter-term. We ignore contributions which are off diagonal in the quarkonia fields since they contribute only at higher orders. The mass splitting is then given by

$$\begin{aligned} \Delta m = & A + B f^2 \left[\frac{1}{(4\pi f)^2} \left((m_\eta)^4 \log \frac{m_\eta^2}{4\pi\mu^2} + 3(m_\pi)^4 \log \frac{m_\pi^2}{4\pi\mu^2} + 4(m_K)^4 \log \frac{m_K^2}{4\pi\mu^2} \right) \right. \\ & \left. - 6C((m_K)^2 + \frac{(m_\pi)^2}{2}) \right] + \frac{D(\mu)}{16\pi^2} (3(m_\pi)^4 + 4(m_K)^4 + (m_\eta)^4) + \dots \end{aligned} \quad (26)$$

The coefficients A , B , C and D are independent of the pion mass. Any pion mass dependence that one might have expected to have arisen from integrating out shorter wavelengths will necessarily have a well defined Taylor expansion about $m_\pi = 0$ and will contribute to a redefinition of D and other higher order chiral symmetry breaking terms. The parameter

f is independent of the pion mass, and for consistency we must know its value up to corrections $O(m_\pi^4)$. At lowest order $f = f_\pi$ and corrections to this relation can be calculated by computing the matrix element of the axial current between the vacuum and the one pion state as is usually done. There will be an unknown counter-term which, in the three flavor theory, can be extracted by measuring the ratio $\frac{f_\pi}{f_K}$ [25]. However, while f is independent of m_π , it is not independent of the number of flavors. Thus, given that all the simulations to date were performed in the two flavor theory, we can not determine the value of f to order m_π^2 . Fortunately, this is unnecessary since all the m_π^2 corrections will go into redefining the parameter D , which we are going to fit anyway. However, since we will not be fitting the piece which is non-analytic in the pion mass we use the relation [26]

$$f = f_\pi \left(1 + \frac{m_\pi^2}{(4\pi f_\pi)^2} \ln \frac{m_\pi^2}{4\pi\mu^2} \right) + \dots \quad (27)$$

Keeping only the leading term in the OPE, we have in addition the constraints on the coefficients:

$$C = 1 \quad (28)$$

and

$$B = \tilde{C}_2 r_B^3 \frac{2g}{\beta(g)} \quad (29)$$

While $C = 1$ is model independent, we see that the numerical value of B depends on the model wavefunction for the onium state.

If we reduce to the case of two families and use the relation (27) we arrive at

$$\Delta m = A_2 - B_2 (2C_2 m_\pi^2 f_\pi^2 + \frac{3m_\pi^4}{16\pi^2} \ln \frac{m_\pi^2}{4\pi\mu^2}) + \frac{D_2(\mu)}{16\pi^2} m_\pi^4. \quad (30)$$

Again, when the first term in the OPE dominates, we have $C_2 = 1$ and $B_2 = \tilde{C}_2 r_B^3 \frac{2g}{\beta(g)}$.

6 Conclusions

We have shown that it is possible to calculate the pion mass dependence of the levels splittings in quarkonia up to a few unknown constants using a chiral Lagrangian. The number of unknown constants which need to be extracted depends upon whether one is willing to

accept the approximation that the leading term in the OPE of the leading multipole result dominates. Within this approximation, the number of unknown parameters can be reduced by mapping the dilatation current in QCD onto the chiral Lagrangian.

We have presented the a full result to order m_π^4 , which could in principle be used not only to calculate errors due to the pion mass, but also errors due to $SU(3)$ breaking, should lattice calculations reach the point where it is possible to calculated with three light quarks. Using the two flavor form of our results we have calculated the errors in the extractions of $\alpha_{\overline{MS}}(M_Z)$ due to the use of unphysical values of the pion mass. Our preliminary results indicate that the errors in the extrapolation are of the same size as the errors quoted in ref. [2]. We emphasize that these results are only preliminary, since we arrived at our numbers using data points for which our expansion in m_π^2 is dubious. We have found however, that under the assumption that the unknown counter-term at order m_π^4 is of the same order as the Log, the expansion is well behaved. Finally, it is very interesting to compare the scale $f_\pi^2 r_B^3$ with the slope determined using the Monte-Carlo data. We find that both numbers are on the order of 10^{-5} MeV^{-1} which we find very encouraging. This number is a factor of 10 smaller than the value determined using Eq. (29) if the model of ref. [5] is used for² \tilde{C}_2 . Therefore, at worst we have underestimated the error in the extrapolation.

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²While we do not trust the model to predict \tilde{C}_2 accurately, we believe the sign, which agrees with the one extracted from the Monte Carlo data, to be trustworthy.

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